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International Journal of Impact Engineering 28 (2003) 363–376

INTERNATIONAL JOURNAL OF IMPACT ENGINEERING

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Extensions to the exact solution of the long-rod penetration/erosion equations

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Received 26 April 2002; received in revised form 7 August 2002

Abstract

The exact solution to the long-rod penetration equations is revisited, in search of improvements to the solution efficiency, while simultaneously enhancing the understanding of the physical parameters that drive the solution. Substantial improvements are offered in these areas. The presentation of the solution is simplified in a way that more tightly unifies the special- and general-case solutions to the problem. Added computational efficiencies are obtained by expressing the general-case solution for penetration and implicit time in terms of a series of Bessel functions. Other extensions and efficiencies are addressed, as well.
Published by Elsevier Science Ltd.

Keywords: Long rods; Penetration; Erosion; Exact solution; Closed-form solution; Terminal rod length; Computational efficiency

1. Background

The penetration equations that describe the behavior of a long rod that erodes while it penetrates at high velocity were formulated independently by Alekseevskii [1] and Tate [2] in the mid-1960s, and are given by

$$L\dot{V} = -Y/\rho_R \quad \text{(rod deceleration)}, \tag{1}$$

$$\frac{1}{2}\rho_R(V-U)^2 + Y = \frac{1}{2}\rho_T U^2 + R \quad \text{(interface stress balance)}, \tag{2}$$

$$V = U - \dot{L} \quad \text{(erosion kinematics)} \tag{3}$$

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0734-743X/02/$ - see front matter Published by Elsevier Science Ltd.
PII: S0734-743X(02)00071-4

and

$$\dot{P} = U \quad \text{(penetration definition)}, \tag{4}$$

where V is the rod velocity, U is the penetration rate, P is the rod penetration, and L is the rod length, all functions of time. The constant parameters include the rod strength Y, the target resistance R, and the target-to-rod density ratio $\gamma = \rho_T/\rho_R$. The dots signify time differentiation. These equations have typically been integrated numerically to achieve a solution. Walters and Segletes [3] obtained an exact solution to these equations. However, the solution was not expressed in terms of the primitive variables that appear in the original equations, but rather in terms of a transformation variable, presented without explanation. Little attempt was made to collate variables into an orderly fashion, leaving an incomplete sense for the term groupings that actually drive the solution. While mathematically rigorous, the solution was somewhat cumbersome to use.

This equation set has been re-examined, in search of improvements and extensions to the solution method, as well as improved solution efficiency. A primary hindrance of the original solution was in the evaluation of the rod velocity as a function of time. While this hindrance remains with the current approach, it may be circumvented by choosing an independent variable other than time, in the evaluation of rod erosion. Indeed, it is often more useful to express the solution in terms of, for example, rod velocity, rather than the canonical function-of-time solution.

Though the governing equations (1)–(4) pertain only to the time during which penetration and erosion simultaneously occur, extensions to the original solution [3] are herein provided for the subsequent stage of rigid-body penetration or rigid-target rod erosion. In addition to the general-case solution to the penetration problem being addressed, several special-case conditions, including the cases for which $R = Y$, and $\rho_R = \rho_T$, respectively, will also be addressed. Not considered herein, however, because of their simplicity, are three special cases for which $R = 0$, $Y = 0$, and $R = Y = 0$, respectively. The present method, described subsequently, can be used to describe the $R = 0$ solution up until the moment that rigid-body penetration commences. Subsequent behavior, however, will be governed by Poncelet flow. In the case of both $Y = 0$ and $R = Y = 0$, the solution becomes trivial in that the rod velocity remains constant until the rod is totally consumed, at which point the event ceases. The penetration velocity and rod erosion rate also remain constant for these cases, in accordance with Eqs. (2) and (3). For the case of $R = Y = 0$, the steady-state erosion rates are governed by the Bernoulli equation.

2. Closed-form solution for L(V)

Without delay, we present the solution to the rod erosion equations, which is valid for all cases (special cases [$R = Y$, $\rho_R = \rho_T$] and the general case):

$$\frac{L}{L_0} = \left(\frac{\sqrt{\gamma}U - \dot{L}}{\sqrt{\gamma}U_0 - \dot{L}_0}\right)^{(R/Y-1)\sqrt{\gamma}} \exp\left[\frac{V_0\dot{L}_0 - V\dot{L}}{2Y/\rho_R}\right], \tag{5}$$

where the "0" subscripts signify conditions at the onset of the penetration event. It is worthy to note that while $-\dot{L}$ is the rate of rod erosion, the term $\sqrt{\gamma}U$ would be the rate of rod erosion were the case hydrodynamic (i.e., were $R = Y = 0$).

Eq. (5) was discerned from the following special- and general-case solutions, themselves obtainable from Eqs. (1)–(3), expressed in terms of a single independent variable, V, the rod velocity. When special- and general-case problems are considered, however, the solutions, at first glance, take on different appearances:

$R = Y$:

$$\text{Governing equation}: \quad \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} V\dot{V}^2 = -\frac{Y}{\rho_R}\ddot{V}, \tag{6}$$

$$\frac{L}{L_0} = \exp\left[\frac{-\rho_R\sqrt{\gamma}}{2Y(1+\sqrt{\gamma})}(V_0^2 - V^2)\right]. \tag{7}$$

$\gamma = 1$:

$$\text{Governing equation}: \quad \left(\frac{V}{2} + \frac{R-Y}{\rho_R V}\right)\dot{V}^2 = -\frac{Y}{\rho_R}\ddot{V}, \tag{8}$$

$$\frac{L}{L_0} = \left(\frac{V}{V_0}\right)^{(R/Y-1)} \exp\left[\frac{-\rho_R}{4Y}(V_0^2 - V^2)\right]. \tag{9}$$

General case:

$$\text{Governing Equation}: \quad \frac{1}{1-\gamma}\left(-\gamma V + \sqrt{\gamma V^2 + 2(R-Y)(1-\gamma)/\rho_R}\right)\dot{V}^2 = -\frac{Y}{\rho_R}\ddot{V}, \tag{10}$$

$$\frac{L}{L_0} = \left(\frac{V}{V_0}\cdot\frac{1+\sqrt{1+2(R-Y)(1-\gamma)/(\gamma\rho_R V^2)}}{1+\sqrt{1+2(R-Y)(1-\gamma)/(\gamma\rho_R V_0^2)}}\right)^{(R/Y-1)\sqrt{\gamma}} \exp\left[\frac{-\rho_R\sqrt{\gamma}}{2Y(1+\sqrt{\gamma})}\right.$$
$$\left.\times\left(V_0^2\frac{\sqrt{1+2(R-Y)(1-\gamma)/(\gamma\rho_R V_0^2)}-\sqrt{\gamma}}{1-\sqrt{\gamma}} - V^2\frac{\sqrt{1+2(R-Y)(1-\gamma)/(\gamma\rho_R V^2)}-\sqrt{\gamma}}{1-\sqrt{\gamma}}\right)\right]. \tag{11}$$

However, Eqs. (7), (9) and (11) have been organized and presented in a manner to demonstrate the functional linkage between the special- and general-case solutions. For example, when either $R = Y$ or $\gamma = 1$, the extended square-root terms of Eq. (11) become unity, leading to the simpler (V/V_0) monomial and $(V_0^2 - V^2)$ exponential terms of Eqs. (9) and (7). When $\gamma = 1$, the leading multiplier on the exponential term in Eq. (11) matches that of Eq. (9). Also, when $R = Y$, the exponent on the monomial becomes zero, leading to the form of Eq. (7). While the forms for $U(V)$ and $\dot{L}(V)$, obtainable from Eqs. (2) and/or (3), are vastly different in appearance for the special and general cases, the solutions for $L(V)$ nonetheless all share a common structured form described by Eq. (5).

3. Choice of model variable

While Eqs. (6), (8) and (10) choose to cast the problem in terms of rod velocity and its derivatives, this is by no means the only option. Alternate expressions of the result, given as $L = L(U)$ or $L = L(\dot{L})$ may be obtained as

$$\frac{L}{L_0} = \left(\frac{U}{U_0} \cdot \frac{1+\sqrt{1+2(R-Y)/(\gamma\rho_R U^2)}}{1+\sqrt{1+2(R-Y)/(\gamma\rho_R U_0^2)}} \right)^{(R/Y-1)\sqrt{\gamma}} \exp\left[\frac{-\rho_R\sqrt{\gamma}(1+\sqrt{\gamma})}{2Y} \cdot \times \left(U_0^2 \frac{\sqrt{1+2(R-Y)/(\gamma\rho_R U_0^2)}+\sqrt{\gamma}}{1+\sqrt{\gamma}} - U^2 \frac{\sqrt{1+2(R-Y)/(\gamma\rho_R U^2)}+\sqrt{\gamma}}{1+\sqrt{\gamma}} \right) \right] \tag{12}$$

$$\frac{L}{L_0} = \left(\frac{\dot{L}}{\dot{L}_0} \cdot \frac{1+\sqrt{1-2(R-Y)/(\rho_R \dot{L}^2)}}{1+\sqrt{1-2(R-Y)/(\rho_R \dot{L}_0^2)}} \right)^{(R/Y-1)\sqrt{\gamma}} \exp\left[\frac{-\rho_R(1+\sqrt{\gamma})}{2Y\sqrt{\gamma}} \cdot \times \left(\dot{L}_0^2 \frac{\sqrt{1-2(R-Y)/(\rho_R \dot{L}_0^2)}+\sqrt{\gamma}}{1+\sqrt{\gamma}} - \dot{L}^2 \frac{\sqrt{1-2(R-Y)/(\rho_R \dot{L}^2)}+\sqrt{\gamma}}{1+\sqrt{\gamma}} \right) \right]. \tag{13}$$

4. Model-variable transformation

The complications of having the model variable V, U, or $\dot{L}$ under the square root for the general case of Eqs. (11), (12), or (13), respectively, may be circumvented with the selection of a mathematically more "natural" variable than the velocity V, U, or $\dot{L}$. Looking to Eq. (5) for guidance, success has been found in

$$\Phi = \frac{\sqrt{\gamma}U - \dot{L}}{\sqrt{|\Sigma|}}, \tag{14}$$

where the constant Σ is defined as $2(R-Y)/\rho_R$. The variable Φ is always nonnegative and follows somewhat the trend of rod velocity V (it actually equals $V/\sqrt{|\Sigma|}$ when $\gamma = 1$). Not surprisingly, Φ is also proportional to $\sqrt{z}$, which was the key transformation variable employed in the original derivation [3]. The key benefit to using the Φ transformation is that $\dot{L}$ and U, rather than requiring square root terms when expressed in V, may be expressed more simply in terms of Φ as

$$\dot{L} = -\frac{\sqrt{|\Sigma|}}{2}\left(\Phi + \mathrm{sgn}(\Sigma)\frac{1}{\Phi}\right) \tag{15}$$

and

$$U = \frac{\sqrt{|\Sigma|}}{2\sqrt{\gamma}}\left(\Phi - \mathrm{sgn}(\Sigma)\frac{1}{\Phi}\right), \tag{16}$$

where the signum function, sgn(x), denotes the sign of the argument [$\mathrm{sgn}(x) = x/|x|$ for $x \neq 0$, and $\mathrm{sgn}(x) = 0$ for $x = 0$], in this case the sign of Σ. The rod velocity, V, may also be obtained directly, by substituting these expressions into Eq. (3):

$$V = \frac{\sqrt{|\Sigma|}}{2\sqrt{\gamma}}\left((\sqrt{\gamma}+1)\Phi + \mathrm{sgn}(\Sigma)\frac{(\sqrt{\gamma}-1)}{\Phi}\right). \tag{17}$$

When Φ is used in preference to rod velocity V as the independent variable, the governing equation (5) leads to the following expression:

$$\frac{L}{L_0} = \left(\frac{\Phi}{\Phi_0}\right)^{(R/Y-1)\sqrt{\gamma}} \exp\left\{-\frac{1}{4\sqrt{\gamma}}\left|\frac{R}{Y}-1\right|\left[\left((\sqrt{\gamma}+1)\Phi_0^2 + \frac{(\sqrt{\gamma}-1)}{\Phi_0^2}\right) - \left((\sqrt{\gamma}+1)\Phi^2 + \frac{(\sqrt{\gamma}-1)}{\Phi^2}\right)\right]\right\}. \tag{18}$$

With minimal rearrangement, the variable Φ can be made to appear always in squared form. It is for this reason that Walters and Segletes [3] selected their transformation variable, z, proportional to Φ^2. We will do the same here, though with a different proportionality constant, so that

$$z = \Phi^2\sqrt{\frac{\sqrt{\gamma}+1}{\left|\sqrt{\gamma}-1\right|}}. \tag{19}$$

By doing so, the expression for residual rod length, Eq. (18), becomes

$$\frac{L}{L_0} = (z/z_0)^{(R/Y-1)(2\sqrt{\gamma})} \exp\left\{-\frac{\sqrt{|\gamma-1|}}{4\sqrt{\gamma}}\left|\frac{R}{Y}-1\right|[(z_0 \pm 1/z_0) - (z \pm 1/z)]\right\}, \tag{20}$$

where the conditional operators in Eq. (20) are chosen as "+" for $\gamma > 1$ and "−" for $\gamma < 1$, and

$$\sqrt{z} = \left(\frac{\sqrt{\gamma}+1}{\left|\sqrt{\gamma}-1\right|\Sigma^2}\right)^{1/4}(\sqrt{\gamma}U - \dot{L}) = \frac{\sqrt{\gamma}V + \sqrt{\gamma V^2 + (1-\gamma)\Sigma}}{\left[\left|\sqrt{\gamma}-1\right|(\sqrt{\gamma}+1)^3\Sigma^2\right]^{1/4}}. \tag{21}$$

Like Eq. (11), the result given by Eq. (20) expresses rod length in terms of a single independent variable, in this case z. The advantage of Eq. (20) over Eq. (11) is in removing the model variable from under a radical. The choice of a proportionality constant different from that used in the prior work [3], when defining z, provides a result that reduces the number of constant parameters in the exponent. More importantly, however, the appearance in the exponential of the model variable in the specific form of $(z \pm 1/z)$ will greatly expedite the evaluation of rod penetration, as will be subsequently shown.

5. Penetration

The evaluation of penetration by way of integrating Eq. (4) may be transformed with Eq. (1) to give

$$P = \int_0^t U\,\mathrm{d}t = -\frac{1}{\dot{V}_0}\int_V^{V_0} \frac{L}{L_0} U\,\mathrm{d}V. \tag{22}$$

The particular functional forms for L and U will govern the form of the solution.

5.1. $R = Y$

Penetration may be directly evaluated in closed form for the simple case of $R = Y$, wherein L is given by Eq. (7), and U is proportional to V throughout the penetration event. In this case, the final penetration (given by P_f as U and V approach zero) is

$$R = Y: \quad P_f = \frac{L_0}{\sqrt{\gamma}}\left(1 - \exp\left[\frac{-\rho_R\sqrt{\gamma}}{2Y(1+\sqrt{\gamma})}V_0^2\right]\right). \tag{23}$$

5.2. $\gamma = 1$

For the $\gamma = 1$ special case, where the penetration velocity U is given algebraically by $U = (V - \Sigma/V)/2$, the penetration may be calculated, as per Eq. (22), in closed form if the value of the V exponent in Eq. (9), given as $(R - Y)/Y$, is an even integer (i.e., R/Y is an odd integer). As an alternative, for cases without the appropriate integer exponents, the penetration for the $\gamma = 1$ special case may be evaluated by way of series solution in terms of velocity. One way to achieve this is to express the penetration as

$$\gamma = 1: \quad \frac{P}{L_0} = \left[\sum_{j=0}^{\infty} a_j\left(\frac{\rho_R V_0^2}{4Y}\right)^j\right] - \frac{L}{L_0}\left[\sum_{j=0}^{\infty} a_j\left(\frac{\rho_R V^2}{4Y}\right)^j\right] \tag{24}$$

and match the derivative of P to the terms of U, given by $U = (V - \Sigma/V)/2$. With this approach, one obtains that $a_0 = -1$, $a_1 = 2/(1 + \rho_R\Sigma/4Y)$, and for the remaining terms, $a_j = -a_{j-1}/(j + \rho_R\Sigma/4Y)$. Note that $\rho_R\Sigma/4Y$ equals $(R/Y - 1)/2$. While the series terms alternate in sign, the fact that j is in the denominator of the recursion formula indicates that the rate of convergence for this solution approach should be similar to that for the exponential series.

Perhaps a more forthright approach for the evaluation of penetration for the $\gamma = 1$ special case (and less prone to the precision problems of evaluating an alternating series) is to directly integrate $LU\,\mathrm{d}V$, as per Eq. (22), by initially expanding the exponential term of L into a series, and integrating term by term to the desired level of precision. As before, $\Sigma = 2(R - Y)/\rho_R$. By integrating this expression with respect to V, as per Eq. (22), one may obtain

$\gamma = 1$:

$$\frac{P}{L_0} = \exp\left[\frac{-\rho_R V_0^2}{4Y}\right]\left\{\sum_{j=0}^{\infty}\frac{1}{j!}\frac{j-\frac{\rho_R\Sigma}{4Y}}{j+\frac{\rho_R\Sigma}{4Y}}\left(\frac{\rho_R V_0^2}{4Y}\right)^j - \left(\frac{V^2}{V_0^2}\right)^{\rho_R\Sigma/4Y}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{j-\frac{\rho_R\Sigma}{4Y}}{j+\frac{\rho_R\Sigma}{4Y}}\left(\frac{\rho_R V^2}{4Y}\right)^j\right\}. \tag{25}$$

5.3. General case

In evaluating the penetration for the general case, the solution becomes more complicated but can nonetheless be made more efficient compared to the method presented in the original solution [3]. Efficiencies are achieved in several ways. The use of rod length L in the form of Eq. (20) retains integer-powered polynomials in the exponential term. As such, the series expansion of the exponential, by which the integrals are evaluated, does not require the evaluation of fractionally powered polynomial expansions, as did the original method [3]. But more importantly, by having transformed L into a form where the exponential argument is of the explicit form $c(z \pm 1/z)$, a method may be used to expand the exponential in an efficient way, reducing the expansion of the exponential to power n from a cost of $(n+1)(n+2)/2$ monomial evaluations in z, to one of $2n+1$ evaluations in z.

The equation describing the penetration, Eq. (22), may be reorganized to obtain an expression in terms of the transformation variable, z:

$$P = \int_0^t U\,\mathrm{d}t = \int_{z_0}^{z}\frac{U}{\dot{V}}\frac{\mathrm{d}V}{\mathrm{d}z}\mathrm{d}z = -\frac{1}{\dot{V}_0}\int_z^{z_0}\frac{L}{L_0}U\frac{\mathrm{d}V}{\mathrm{d}z}\mathrm{d}z. \tag{26}$$

Using Eqs. (17) and (19), the rod velocity is expressible in terms of z, so that $\mathrm{d}V/\mathrm{d}z$ may be computed as

$$\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{(|\gamma-1|\Sigma^2)^{1/4}}{4\sqrt{\gamma}}\left[\frac{(\sqrt{\gamma}+1)^{1/2}}{z^{1/2}} - \mathrm{sgn}[(\gamma-1)\Sigma]\frac{\left|\sqrt{\gamma}-1\right|^{1/2}}{z^{3/2}}\right]. \tag{27}$$

In a similar vein, from Eqs. (16) and (19), U may be expressed in terms of z, and the product, $U\,\mathrm{d}V/\mathrm{d}z$, may therefore be computed as

$$U\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{|\gamma-1|^{1/4}|\Sigma|}{8\gamma}\left\{|\gamma-1|^{1/4} - \mathrm{sgn}(\Sigma)\left(\left[\frac{(\sqrt{\gamma}+1)^3}{\left|\sqrt{\gamma}-1\right|}\right]^{1/4} + \mathrm{sgn}(\gamma-1)\left[\frac{\left|\sqrt{\gamma}-1\right|^3}{(\sqrt{\gamma}+1)}\right]^{1/4}\right)\frac{1}{z} + \mathrm{sgn}(\gamma-1)\frac{|\gamma-1|^{1/4}}{z^2}\right\}, \tag{28}$$

which is of the form

$$U\,\mathrm{d}V/\mathrm{d}z = A(a_0 + a_1/z + a_2/z^2). \tag{29}$$

Substituting this result and the transformed expression for L, given by Eq. (20), into Eq. (26) allows the integral for penetration to take the form

$$P = B_P \int_z^{z_0} (a_0 + a_1/z + a_2/z^2) z^b \exp[c(z \pm 1/z)]\,\mathrm{d}z, \tag{30}$$

where the conditional minus sign in the exponential is taken when $\gamma < 1$, and a_i, b, c, and B_P are all constants, expressible as

$$a_0 = |\gamma - 1|^{1/4}, \tag{31}$$

$$a_1 = -\mathrm{sgn}(R - Y)\left(\left[\frac{(\sqrt{\gamma}+1)^3}{\left|\sqrt{\gamma}-1\right|}\right]^{1/4} + \mathrm{sgn}(\gamma - 1)\left[\frac{\left|\sqrt{\gamma}-1\right|^3}{(\sqrt{\gamma}+1)}\right]^{1/4}\right), \tag{32}$$

$$a_2 = \mathrm{sgn}(\gamma - 1)|\gamma - 1|^{1/4}, \tag{33}$$

$$b = \frac{1}{2\sqrt{\gamma}}\left(\frac{R}{Y} - 1\right), \tag{34}$$

$$c = \frac{1}{4}\sqrt{\frac{|\gamma - 1|}{\gamma}}\left|\frac{R}{Y} - 1\right| \tag{35}$$

and

$$B_P = L_0\left|\frac{R}{Y} - 1\right|\frac{|\gamma - 1|^{1/4}}{4\gamma z_0^b}\exp[-c(z_0 + \mathrm{sgn}(\gamma - 1)/z_0)]. \tag{36}$$

The form of Eq. (30) is basically identical to an intermediate step of the original solution [3], though with differently defined constants. The prior work [3] opted to transform the equation again to eliminate the leading polynomials, but did so at the expense of introducing non-integer powers into the exponential term. Then, the penetration equation was solved by expanding the exponential into a power series of $(A_1 z^g + A_2 z^{-g})^j$ terms and expanding each $(A_1 z^g + A_2 z^{-g})^j$ term into $j+1$ monomials, using a binomial expansion. The net result of the total expansion was that, to include terms out to a power of $j = n$, a total of $(n+1)(n+2)/2$ monomials were generated, and then integrated term by term. With n routinely exceeding 20 to obtain the desired precision, and approaching 100 for certain initial conditions, the computational burden was substantial, though still more efficient than a numerical integration of Eqs. (1)–(4).

While the currently proposed method still relies on a series expansion of the exponential to perform the integration, a technique permits a streamlined method for achieving the expansion. In particular, a method exists to expand the subject exponential series with the form

$$\exp[c(z \pm 1/z)] = \sum_{j=-\infty}^{\infty} C_j^{\pm} z^j, \tag{37}$$

where the C_j^+ or C_j^- coefficients are a function only of the parameter c. In particular, the C_j^- constants are given by evaluations of Bessel functions of the first kind, such that $C_j^- = J_j(2c)$. The C_j^+ constants, by contrast, are given by modified Bessel functions of the first kind, such that $C_j^+ = I_j(2c)$. The expansion using the form of Eq. (37), to include terms of power $z^{\pm n}$, requires the evaluation of only $2n+1$ monomials in z, and therefore represents a significant improvement over the method previously employed [3], which required the evaluation of $(n+1)(n+2)/2$ monomials in z for identical precision.

While there is an overhead associated with the evaluation of the $C_j^{\pm}$ parameters, given by the converging series that defines the Bessel functions for integer order:

$$C_j^{\pm} = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{(\pm 1)^i c^{2i+j}}{i!(i+j)!}, & j \geqslant 0 \\ (\pm 1)^j C_{-j}^{\pm}, & j < 0, \end{cases} \tag{38}$$

the parameter c is fixed by the initial conditions (material properties) of the penetration problem. As such, the C_j^+ or C_j^- terms may be calculated once at the onset of the analysis, regardless of how many z values (i.e., velocities) for which the solution needs evaluation. Furthermore, there exists a recursive technique for evaluating the $C_j^{\pm}$ parameters of Eq. (38), based on the recursions

$$\frac{C_j^+}{C_{j-1}^+} = \frac{1}{\dfrac{j}{c} + \dfrac{C_{j+1}^+}{C_j^+}} \tag{39a}$$

and

$$\frac{C_j^-}{C_{j-1}^-} = \frac{1}{\dfrac{j}{c} - \dfrac{C_{j+1}^-}{C_j^-}}, \tag{39b}$$

which thereby offer further computational savings.

The integration for penetration is, thus, finally achieved by employing this optimized expansion and integrating term by term and evaluating at the desired limits. When b is not an integer, which is the typical case, the result may be expressed as

$$\text{General case :} \quad P = B_P \sum_{j=-\infty}^{\infty} (a_0 C_{j-1}^{\pm} + a_1 C_j^{\pm} + a_2 C_{j+1}^{\pm}) \left. \frac{z^{j+b}}{j+b} \right|_{z}^{z_0}, \tag{40}$$

where the C^+ terms are used when $\gamma > 1$ and the C^- terms are used when $\gamma < 1$. For the case when b is an integer, the single term of Eq. (40) that would otherwise produce a zero in the denominator (i.e., the term for which $j = -b$) originated from a $1/z$ integration, and would actually have produced, upon integration, the logarithmic term $\ln(z)$, instead of $z^{j+b}/(j+b)$.

6. Implicit time

Though these solutions for $L(V)$ and $P(V)$ bypass the intermediate evaluation of $V(t)$, the penetration variables may, if needed, be implicitly expressed in terms of time, by integration of $L(V)$,

$$t = \int_{V_0}^{V} \frac{\mathrm{d}V}{\dot{V}} = -\frac{1}{\dot{V}_0}\int_{V}^{V_0} \frac{L}{L_0}\,\mathrm{d}V, \tag{41}$$

in order to obtain $t(V)$. As in the case of penetration, a closed-form solution to Eq. (41) will be possible only for the special case of $\gamma = 1$, and then only when $(R - Y)/Y$ is an odd integer (i.e., R/Y is even). In all other cases, the integration of Eq. (41) will take the form of a series solution. Of the several ways to obtain a series integration of the special case solutions, a power-series expansion is preferable to a repeated integration-by-parts solution because it avoids an alternating series, for the case when the "c" constant associated with the $\exp[-c(V_0^2 - V^2)]$ term is positive. Such is always the case for penetration problems. Thus, the special-case solutions for $t(V)$ may be evaluated as

$R = Y$:

$$t = \frac{\rho_{\mathrm{R}} L_0 V_0}{Y}\exp\left[\frac{-\rho_{\mathrm{R}}\sqrt{\gamma}V_0^2}{2Y(1+\sqrt{\gamma})}\right]\left[\sum_{i=0}^{\infty}\frac{1}{i!(2i+1)}\left(\frac{\rho_{\mathrm{R}}\sqrt{\gamma}V_0^2}{2Y(1+\sqrt{\gamma})}\right)^i - \frac{V}{V_0}\sum_{i=0}^{\infty}\frac{1}{i!(2i+1)}\left(\frac{\rho_{\mathrm{R}}\sqrt{\gamma}V^2}{2Y(1+\sqrt{\gamma})}\right)^i\right]. \tag{42}$$

$\gamma = 1$:

$$t = \frac{\rho_{\mathrm{R}} L_0 V_0}{Y}\exp\left[\frac{-\rho_{\mathrm{R}} V_0^2}{4Y}\right]\left[\sum_{i=0}^{\infty}\frac{1}{i!(2i+R/Y)}\left(\frac{\rho_R V_0^2}{4Y}\right)^i - \left(\frac{V}{V_0}\right)^{R/Y} \times \sum_{i=0}^{\infty}\frac{1}{i!(2i+R/Y)}\left(\frac{\rho_{\mathrm{R}} V^2}{4Y}\right)^i\right]. \tag{43}$$

For the general case, a solution is most profitably obtained in a manner analogous to the penetration evaluation, in which a transformation to z facilitates a streamlined series solution:

$$t = \int_{V_0}^{V} \frac{\mathrm{d}V}{\dot{V}} = -\frac{1}{\dot{V}_0}\int_{z}^{z_0} \frac{L}{L_0}\frac{\mathrm{d}V}{\mathrm{d}z}\,\mathrm{d}z. \tag{44}$$

This integration may be staged through the substitution of Eqs. (20) and (27), to give the following form:

$$t = B_t\int_{z}^{z_0}(d_0/z^{1/2} + d_1/z^{3/2})z^b\exp[c(z \pm 1/z)]\,\mathrm{d}z, \tag{45}$$

where the conditional minus sign is taken when $\gamma < 1$. Here, b and c are defined as before, by Eqs. (34) and (35), while

$$d_0 = (\sqrt{\gamma}+1)^{1/2}, \tag{46}$$

$$d_1 = -\mathrm{sgn}[(\gamma-1)(R-Y)]\left|\sqrt{\gamma}-1\right|^{1/2} \tag{47}$$

and

$$B_t = L_0\sqrt{\frac{\rho_R}{Y}}\left(\frac{1}{8\gamma}\left|\frac{R}{Y}-1\right|\right)^{1/2}\frac{|\gamma-1|^{1/4}}{z_0^b}\exp[-c(z_0+\mathrm{sgn}(\gamma-1)/z_0)]. \tag{48}$$

By using a method analogous to that in Eqs. (37)–(40), and with the same definitions for $C_j^{\pm}$ [given by Eq. (38), where the "+" solution applies for $\gamma > 1$, and the "−" solution for $\gamma < 1$], the expression for t given by Eq. (45) may be expanded in a series as

$$\text{General case}: \quad t = B_t\sum_{j=-\infty}^{\infty}(d_0C_{j-1}^{\pm}+d_1C_j^{\pm})\left.\frac{z^{j+b-1/2}}{j+b-1/2}\right|_z^{z_0}. \tag{49}$$

Like Eq. (40), there is one exception to the general validity of this result, specifically for the case when b is precisely a half integer. If and only if this is the case, a single term of Eq. (49) will require modification: namely, the term for which $j+b-1/2$ exactly equals zero, originating from a $1/z$ integration. This integration would, for this one term only, rightfully have produced a $\ln(z)$ term, instead of $z^{j+b-1/2}/(j+b-1/2)$. As with the evaluation of penetration, the summation of Eq. (49) is carried out for j over some finite range from $-n$ to $+n$ so as to achieve the desired level of precision.

7. Terminal rod length

The "terminal" rod length may be ascertained for the various solution cases [from Eqs. (7), (9) or (11)], by setting V to its terminal value, $V_x = \sqrt{\Sigma}$ for the case of $R > Y$ and $V_x = \sqrt{-\Sigma/\gamma}$ for $R < Y$, with the parameter Σ given by $\Sigma = 2(R-Y)/\rho_R$. When $R > Y$, this termination corresponds to the point where $U = 0$, when the penetration ceases (though the rod may continue to erode thereafter). For $R < Y$, the termination corresponds to the point where $\dot{L} = 0$, when the rod erosion ceases (though the rod may continue to penetrate as a rigid body thereafter). This terminal state, denoted with the subscript "x", corresponds not to the end of the ballistic event, but rather to the time at which the governing Eqs. (1)–(4) cease to apply. In those governing equations, developed for the case of a simultaneously eroding rod and target, the subscript "x" condition corresponds to the moment at which either the rod or the target stops eroding. In general, these two conditions do not occur simultaneously. The rod length (normalized) at the terminal state "x" for the various cases are expressible as

$$R = Y: \quad \frac{L_x}{L_0} = \exp\left[\frac{-\rho_R\sqrt{\gamma}}{2Y(1+\sqrt{\gamma})}V_0^2\right], \tag{50}$$

$$\gamma = 1: \quad \frac{L_x}{L_0} = \left[\frac{V_0^2}{|\Sigma|}\right]^{-(R/Y-1)/2}\exp\left[-\frac{1}{2}\left|\frac{R}{Y}-1\right|\left(\frac{V_0^2}{|\Sigma|}-1\right)\right], \tag{51}$$

General case:

$$\frac{L_x}{L_0} = \left(\frac{\sqrt{\gamma}(V_0/\sqrt{|\Sigma|}) + \sqrt{\gamma(V_0/\sqrt{|\Sigma|})^2 + \mathrm{sgn}(\Sigma)(1-\gamma)}}{1+\sqrt{\gamma}}\right)^{-(R/Y-1)/\sqrt{\gamma}} \exp\left[-\left|\frac{R}{Y}-1\right| \cdot \times \left(\frac{(V_0/\sqrt{|\Sigma|})\sqrt{\gamma(V_0/\sqrt{|\Sigma|})^2 + \mathrm{sgn}(\Sigma)(1-\gamma)} - \gamma(V_0/\sqrt{|\Sigma|})^2}{1-\gamma} - \frac{1+\mathrm{sgn}(\Sigma)}{2}\right)\right]. \tag{52}$$

For cases where $R > Y$, this terminal length corresponds to that length of rod as of the moment that penetration ceases. For $R < Y$, this is the rod length at the onset of rigid-body penetration.

8. Residual erosion and penetration behavior

Eqs. (1) and (2) are valid only while there is simultaneous target penetration and rod erosion. Except for the special case of $R = Y$, $\dot{L}$ and U will not simultaneously approach zero. In the general case then, the physical event will continue with either residual rod erosion following the cessation of penetration (when $R > Y$) or residual rigid body penetration following the cessation of rod erosion (when $R < Y$). These afterflow events are amenable to closed-form analytical solution. Continuing to denote the state at this transition point (the moment of transition to either rigid target or rigid rod) with the use of the subscript "x", the absolute final state, when the rod velocity itself finally reaches zero, will be denoted with the subscript "f ". Recall that $V_x = \sqrt{\Sigma}$ when $R > Y$, while $V_x = \sqrt{-\Sigma/\gamma}$ when $R < Y$, where $\Sigma = 2(R - Y)/\rho_R$.

8.1. Residual rod erosion

For the case of $R > Y$, the target becomes rigid while rod erosion continues. To deal with this, Eq. (2) is replaced by the constraint $U = 0$. The kinematic constraint of Eq. (3) becomes, as a result, $V = -\dot{L}$. Solving Eq. (1) for L, differentiating, and substituting the revised kinematic constraint to eliminate $\dot{L}$, one obtains as the governing equation

$$V\dot{V}^2 = -(Y/\rho_R)\ddot{V}. \tag{53}$$

The result (as a function of V) is that

$$L = L_x \exp\left[\frac{-\rho_R}{2Y}(V_x^2 - V^2)\right]. \tag{54}$$

Evaluating the penetration and rod length at the final state (where $V = 0$), one obtains $P_f = P_x$ and

$$L_f = L_x \exp\left[\frac{-\rho_R V_x^2}{2Y}\right]. \tag{55}$$

Because of the similarity between the governing equation here, Eq. (53), and the special case $R = Y$ governing equation, Eq. (6), the duration of this residual-erosion phase of the rod may likewise be calculated with the same power-series-solution form used to calculate event duration for the special cases. Use of this form leads to

$$t - t_x = \frac{\rho_R L_x V_x}{Y} \exp\left(\frac{-\rho_R V_x^2}{2Y}\right) \left[\sum_{i=0}^{\infty} \frac{1}{i!(2i+1)} \left(\frac{\rho_R V_x^2}{2Y}\right)^i - \frac{V}{V_x} \sum_{i=0}^{\infty} \frac{1}{i!(2i+1)} \left(\frac{\rho_R V^2}{2Y}\right)^i\right], \quad (56)$$

which, as V approaches zero, becomes the following result:

$$t_f - t_x = \frac{\rho_R L_x V_x}{Y} \exp\left(\frac{-\rho_R V_x^2}{2Y}\right) \sum_{i=0}^{\infty} \frac{1}{i!(2i+1)} \left(\frac{\rho_R V_x^2}{2Y}\right)^i. \quad (57)$$

8.2. Residual rigid-body penetration

For the alternate case of $R < Y$, a state of rigid-body penetration is reached after the rod erosion ceases. As before, Eq. (2) is replaced by the constraint $\dot{L} = 0$. The kinematic constraint (3) becomes, as a result, $V = U$. However, there is one additional modification required for the governing equations. In particular, the force causing the rod deceleration in Eq. (1) is no longer Y, since the rod is no longer in a plastic state. Rather, it is a diminished stress state applied by the pressure head and resistance of the target, $1/2\rho_T U^2 + R$. But since, kinematically, $V = U$ and L remains fixed at L_x, the rod deceleration equation becomes

$$L_x \dot{V} = -(1/2\rho_T V^2 + R)/\rho_R. \quad (58)$$

This may be solved as

$$V = U = \sqrt{\frac{2R}{\rho_T}} \tan\left[\frac{\gamma}{L_x}\sqrt{\frac{R}{2\rho_T}}(t_x - t) + \tan^{-1}\left(V_x\sqrt{\frac{\rho_T}{2R}}\right)\right]. \quad (59)$$

The final time, at which the velocity drops to zero, is found to be

$$t_f = t_x + \frac{L_x}{\gamma}\sqrt{\frac{2\rho_T}{R}} \tan^{-1}\left(V_x\sqrt{\frac{\rho_T}{2R}}\right). \quad (60)$$

The expression for U, which is Eq. (59), may be integrated one more time to obtain the differential penetration that occurs during the afterflow phase. One obtains

$$P - P_x = \frac{2L_x}{\gamma}\left\{\log\cos\left[\frac{\gamma}{L_x}\sqrt{\frac{R}{2\rho_T}}(t_f - t)\right] - \log\cos\tan^{-1}\left(V_x\sqrt{\frac{\rho_T}{2R}}\right)\right\}. \quad (61)$$

When evaluated at $t = t_f$, and employing some trigonometric substitutions, the final result is that $L_f = L_x$ and the afterflow penetration is

$$P_f - P_x = \frac{L_x}{\gamma}\log\left(1 + \frac{\rho_T V_x^2}{2R}\right). \quad (62)$$

9. Conclusions

This report presents updated results related to the exact solution of the long-rod penetration equations, formulated by Alekseevskii [1] and Tate [2], and first solved by Walters and Segletes [3].

While the original solution [3] is accurate and comprehensive, there have been a number of improvements or enhancements, both to the presentation and the solution approach.

Eq. (5) is a concise analytical presentation of rod length as a function of rod velocity, valid for both special and general cases, providing an enhanced sense for the terms that drive the analytical solution. Eqs. (6)–(11) compare and contrast the special- and general-case analytical solutions, while Eqs. (12) and (13) present the result in terms of an alternate model variable. The key independent variable transformation (to z), unexplained but indispensable to the original solution, is herein developed more fully and much of its mystery is thereby uncloaked. Further, its expression is slightly altered from the original solution, resulting, by comparison, in a form amenable to a highly streamlined series solution for penetration $P(z)$, as Eq. (40), or implicit time, $t(z)$, as Eq. (49). Extensions are presented to the original solution, which account for the period of rigid-body penetration or rigid-target rod erosion that follows the period of eroding-body penetration addressed by the original penetration equations.

While based on the original solution of Walters and Segletes [3], the current work offers enhanced appreciation and understanding of the original effort, as well as extensions to the original work. Finally, the streamlined techniques presented herein make any implementation of the solution significantly more efficient than the original solution technique.

References

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