23RD INTERNATIONAL SYMPOSIUM ON BALLISTICS
TARRAGONA, SPAIN 16-20 APRIL 2007

SIZE DISTRIBUTION OF FRAGMENTS GENERATED BY DETONATION OF FRAGMENTING WARHEADS

Predrag Elek and Slobodan Jaramaz

University of Belgrade, Faculty of Mechanical Engineering
Weapon Systems Department, Kraljice Marije 16, 11120 Belgrade 35, Serbia

The paper considers statistical aspects of high explosive warhead fragmentation. Modeling of fragment size (mass) distribution is of the great importance for determination of fragmenting warhead efficiency. Seven relevant theoretical fragmentation mass distribution models are reviewed: the Mott, the generalized Mott, the Grady, the generalized Grady, the lognormal, the Weibull and the Held distribution. Comparison of these models with representative experimental database of 30 fragmenting projectiles has shown generally very good correspondence between theoretical models and experimental data. The analysis has indicated that the generalized Mott, the generalized Grady and the Weibull distribution enable the best description of experimental fragment mass distribution data. The generalized Grady distribution has somewhat better results, and its bimodal characteristic can be physically justified. Suggested theoretical fragment mass distribution laws will be applied in a complex fragmenting projectile efficiency simulation model.

INTRODUCTION

Modeling of fragmentation process is of the utmost importance for design, redesign and efficiency analysis of HE projectiles. The fragment mass distribution along with the initial fragment velocity, the spatial and the shape distribution of fragments, enables complete characterization of a fragmentation process. Fragmentation of HE projectile is the result of complex processes of explosive detonation, gas products expansion and behavior of the casing material under the intensive impulse loads. Final character and distribution of cracks in the projectile casing determinate shape, size and mass of formed fragments. There are several approaches to the fragmentation problem – probabilistic, energetic, approach based on fracture mechanics, etc. Having in mind the complexity of underlying physics, the statistical analysis based on experimental results seems to be a promising approach to the fragment mass distribution problem.

FRAGMENT MASS DISTRIBUTION LAWS

Fragment mass distribution is usually described by a cumulative distribution function, rather than a probability density function (histogram), which is more sensitive to the scatter of the fragment masses data. The cumulative number of fragments $N_T(m)=N_T(>m)$ is the total number of fragments with mass greater than m, and alternatively, the cumulative fragment mass $M_T(m)=M_T(>m)$ is the total mass of all fragments with individual mass greater than m.

In this paper, relative (normalized) cumulative distributions $N(m)$ and $M(m)$ will be used

$$N(m) = \frac{N_T(m)}{N_0}, \qquad M(m) = \frac{M_T(m)}{M_0}, \tag{1}$$

where N_0 is the total fragment number and M_0 is the total mass of fragments. The relation between the cumulative distributions is

$$\frac{\mathrm{d}M}{\mathrm{d}m} = \frac{m}{\bar{m}} \frac{\mathrm{d}N}{\mathrm{d}m}. \tag{2}$$

The average fragment mass (distribution mean), which is the most important characteristic of the distribution, is determined by

$$\bar{m} = \frac{M_0}{N_0} = \int_0^\infty N(m)\mathrm{d}m . \tag{3}$$

Very useful numerical property of the distribution is the median, which is defined by

$$N(\tilde{m}_N) = \tfrac{1}{2}, \text{ and } M(\tilde{m}_M) = \tfrac{1}{2}. \tag{4}$$

There are numerous distribution laws that are used to describe a real distribution of HE projectile fragments. The most relevant distribution laws will be outlined here.

Mott distribution. In his classis works [1], based on two-dimensional geometric statistics, Mott had formulated the well-known fragment distribution law in the form

$$N(m) = \mathrm{e}^{-(\frac{m}{\mu})^{\frac{1}{2}}} . \tag{5}$$

Generalized Mott distribution. Mott had argued that in three-dimensional fragmentation of thick-walled cylinder, where fragments do not retain the inner and outer surface of original cylinder, exponent ⅓ instead ½ in eq. (5) would be more appropriate. Introducing exponent λ in eq. (5), we get the generalized Mott distribution (e.g. [2], [3]) as

$$N(m) = \mathrm{e}^{-\left(\frac{m}{\mu}\right)^{\lambda}}. \tag{6}$$

This distribution corresponds to the two-parametric Weibull distribution.

Grady distribution. Following Mott's approach based on a Poisson distribution of fracture points, Grady [4] established an alternative paradigm, defined also in [5], and proposed the simple linear exponential distribution

$$N(m) = \mathrm{e}^{-\frac{m}{\mu}}. \tag{7}$$

This distribution law is a special case of the generalized Mott distribution law (eq. (6)), for λ=1.

Generalized Grady distribution. Considering statistically inhomogeneous fragmentation, Grady [4] analyzed the three-parametric generalization of distribution defined by eq. (7) as follows:

$$N(m) = f\mathrm{e}^{-\frac{m}{\mu_1}} + \left(1-f\right)\mathrm{e}^{-\frac{m}{\mu_2}}. \tag{8}$$

Lognormal distribution. Observing multiplicative nature of fragmentation process, several authors (e.g. [6]) suggested the lognormal distribution for describing the fragment mass distribution:

$$N(m) = \tfrac{1}{2}\left[1 - \mathrm{erf}\left(\tfrac{\ln m - \mu}{\sqrt{2}\sigma}\right)\right], \tag{9}$$

where erf is the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t$.

Weibull distribution. The two-parametric Weibull distribution (also known as the Rosin-Ramler distribution), originally used for the description of the grain size distribution in grinding processes, defines the normalized cumulative mass as

$$M(m) = \mathrm{e}^{-\left(\frac{m}{\mu}\right)^{\lambda}}. \tag{10}$$

Using eq. (2), one gets the relative cumulative number of fragments:

$$N(m) = \frac{1}{\Gamma\left(1-\frac{1}{\lambda}\right)}\Gamma\left(1-\tfrac{1}{\lambda},\left(\tfrac{m}{\mu}\right)^{\lambda}\right),\ \lambda > 1. \tag{11}$$

In eq. (11), $\Gamma(a, x)$ is the upper incomplete gamma function $\Gamma(a,x) = \int_x^{\infty} t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$.

Held distribution. Held [7] had presented the relation between the cumulative mass and the cumulative fragment number in the form

$$M(n) = M_0(1 - \mathrm{e}^{-Bn^{\lambda}}), \tag{12}$$

where n is the cumulative number of fragments sorted in descending order, and $M(n)$ is the total mass of these fragments. The mass of each particular fragment can be calculated by

$$m_n = M(n) - M(n-1). \tag{13}$$

Transformation of eq. (12), using relation (2) leads to the implicit form of the cumulative number distribution:

$$m = M_0 B\lambda N_T^{\lambda-1}(m)\mathrm{e}^{-BN_T^{\lambda-1}(m)}. \tag{14}$$

The review of fragment mass distribution laws is given in Table 1. The medians can be easily calculated form eq. (4).

Table 1. Fragment mass distribution laws and their properties

Distribution	**Relative cumulative number of fragments**	**Relative cumulative mass of fragments**	**Distribution mean**
	N(m)=N(>m)	*M(m)=M(>m)*	$\overline{m}$
Mott	$\mathrm{e}^{-\left(\frac{m}{\mu}\right)^{\frac{1}{2}}}$	$\frac{1}{2}\Gamma\left(3,\left(\frac{m}{\mu}\right)^{\frac{1}{2}}\right)$	2μ
Generalized Mott	$\mathrm{e}^{-\left(\frac{m}{\mu}\right)^{\lambda}}$	$\frac{1}{\Gamma\left(1+\frac{1}{\lambda}\right)}\Gamma\left(1+\frac{1}{\lambda},\left(\frac{m}{\mu}\right)^{\lambda}\right)$	$\Gamma\left(1+\frac{1}{\lambda}\right)\mu$
Grady	$\mathrm{e}^{-\frac{m}{\mu}}$	$\left(1+\frac{m}{\mu}\right)\mathrm{e}^{-\frac{m}{\mu}}$	μ
Generalized Grady	$f\mathrm{e}^{-\frac{m}{\mu_1}} + (1-f)\mathrm{e}^{-\frac{m}{\mu_2}}$	$\frac{1}{\overline{m}}\left[f(\mu_1+m)\mathrm{e}^{-\frac{m}{\mu_1}} + (1-f)(\mu_2+m)\mathrm{e}^{-\frac{m}{\mu_2}}\right]$	$f\mu_1 + (1-f)\mu_2$
Log-normal	$\frac{1}{2}\left[1-\mathrm{erf}\left(\frac{\ln m-\mu}{\sqrt{2}\sigma}\right)\right]$	$\frac{1}{2}\left[1-\mathrm{erf}\left(\frac{\ln m-(\mu+\sigma^2)}{\sqrt{2}\sigma}\right)\right]$	$\mathrm{e}^{\mu+\sigma^2/2}$
Weibull	$\frac{1}{\Gamma\left(1-\frac{1}{\lambda}\right)}\Gamma\left(1-\frac{1}{\lambda},\left(\frac{m}{\mu}\right)^{\lambda}\right),\ \lambda>1$	$\mathrm{e}^{-\left(\frac{m}{\mu}\right)^{\lambda}}$	$\frac{\mu}{\Gamma\left(1-\frac{1}{\lambda}\right)}$
Held	$M(m) = 1-\mathrm{e}^{-BN_T^{\lambda}(m)},\ m = M_0 B\lambda N_T^{\lambda-1}(m)\mathrm{e}^{-BN_T^{\lambda-1}(m)}$		n/a

In the earlier paper [8], it had been shown that the Strømsøe-Ingebrigtsen distribution [9] does not represent substantial improvement of the Mott law. It is also discussed that the widely applicable power-law distribution (e.g. [10]) cannot successfully describe HE projectile fragmentation. Finally, the Gilvarry distribution [11] has not been analyzed here, regarding a four-parameter fit impractical.

COMPARISON WITH EXPRIMENTS AND DISCUSSION

In order to validate presented theoretical distribution models, a comparison with experimental data has been performed. It has been used experimental data from [12] (20 projectiles), [7] (3 projectiles), [9] (3 projectiles), [3] (2 projectiles) and [5] (2 projectiles). Although in some fragmentation analysis the fragments with the greatest masses are neglected (supposed as the result of irregular fragmentation), all collected fragments has been taken into account in this research.

The parameters in the theoretical distribution laws are calculated by minimizing the deviation of the theoretical from the real distribution in the sense of the least squares method. Characteristic diagrams of experimental data and the corresponding theoretical models for typical experimental projectile are given in Figure 1.

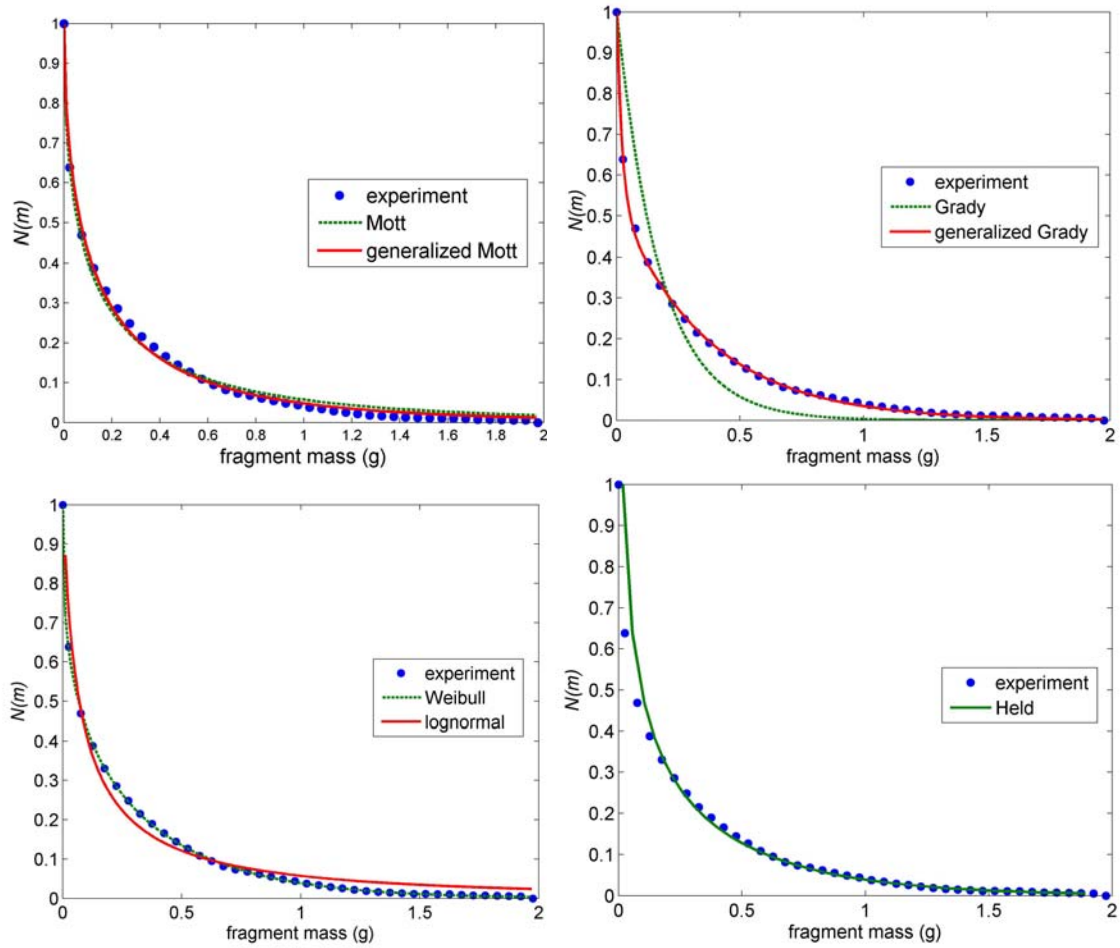

Figure 1. Comparison of the relative cumulative fragment number experimental data for projectile 1 [12] with: a) the Mott and the generalized Mott, b) the Grady and the generalized Grady, c) the Weibull and the lognormal and d) the Held fragment mass distribution model

In order to evaluate and compare the goodness of fits, the degrees of freedom adjusted coefficient of determination is used:

$$\bar{R}^2 = 1 - \frac{S_D}{S_T}\frac{n-1}{n-k-1}, \tag{15}$$

where S_D and S_T are

$$S_D = \sum_{j=1}^{n}(F_t(m_j) - F_e(m_j))^2, \quad S_T = \sum_{j=1}^{n}(F_t(m_j) - \bar{F}_e)^2 . \tag{16}$$

In eq. (16), $\bar{F}_e$ is the mean of an experimental distribution, n is the number of mass groups and k is the number of adjustable parameters in a distribution law.

The survey of coefficient of determination values for the analyzed theoretical distribution models applied on experimental data for 30 projectiles is given in Table 1 (target function is $N(m)$). Generally, all models (except the Grady's) are good approximation of experimental data, but the generalized Mott, the generalized Grady and the Weibull distribution have the highest coefficients of determination. Similar results are obtained from the cumulative fragment mass $M(m)$ optimization (Figure 2), slightly favouring the generalized Grady distribution. This means that fragments have two characteristic sizes (masses), which is shown in Figure 3. Physically, finer and coarser fragments can be related to the central cylindrical and the residual portion of the projectile. Another explanation is that different fragment formation mechanisms in the inner and the outer section of the projectile casing influence the bimodal distribution. The generalized Grady distribution predicts the median $\tilde{m}_M$ well, but the mean $\bar{m}$ and the median $\tilde{m}_N$ systematically underestimates the experimental values. The reason could be unreliable number of fine fragments in the smallest mass group, i.e. the mass of fragments that are not recovered.

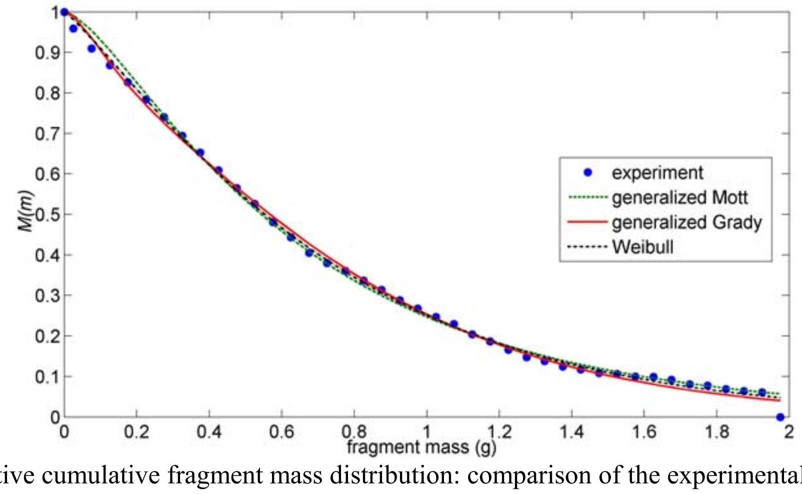

Figure 2. Relative cumulative fragment mass distribution: comparison of the experimental data (projectile 1 from [12]) and the generalized Mott, the generalized Grady and the Weibull distribution

Table 2. The coefficient of determination calculated for different fragment mass distribution laws and 30 experiments. Relative cumulative fragment number $N(m)$ has been fitted

	Coefficient of determination						
	Mott	***gen. Mott***	***Grady***	***gen. Grady***	***Lognormal***	***Weibull***	***Held***
1	0.9919	0.9960	0.9271	0.9994	0.9817	0.9997	0.9975
2	0.9946	0.9946	0.9160	0.9999	0.9846	0.9995	0.9899
3	0.9936	0.9936	0.9025	0.9992	0.9787	0.9997	0.9981
4	0.9948	0.9957	0.9252	0.9995	0.9897	0.9994	0.9944
5	0.9896	0.9913	0.8720	0.9994	0.9766	0.9993	0.9988
6	0.9122	0.9951	0.9909	0.9987	0.9807	0.9980	0.9926
7	0.4326	0.9462	0.9026	0.8959	0.9229	0.9521	0.9687
8	0.9984	0.9985	0.9337	0.9973	0.9937	0.9996	0.9606
9	0.9952	0.9984	0.9222	0.9970	0.9946	0.9994	0.9723
10	0.9941	0.9984	0.9120	0.9958	0.9941	0.9994	0.9855
11	0.9990	0.9992	0.9521	0.9985	0.9961	0.9990	0.9805
12	0.9991	0.9991	0.9554	0.9994	0.9964	0.9995	0.9775
13	0.9750	0.9988	0.9599	0.9997	0.9909	0.9961	0.9885
14	0.9884	0.9996	0.9521	0.9990	0.9931	0.9959	0.9967
15	0.9995	0.9997	0.9848	0.9998	0.9990	0.9999	0.9853
16	0.9986	0.9986	0.9667	0.9995	0.9964	0.9999	0.9812
17	0.8879	0.9990	0.9985	0.9990	0.9940	0.9972	0.9760
18	0.9959	0.9971	0.8911	0.9943	0.9951	0.9931	0.9908
19	0.9995	0.9997	0.9553	0.9999	0.9975	0.9991	0.9944
20	0.9990	0.9992	0.9461	0.9996	0.9963	0.9998	0.9899
21	0.9923	0.9996	0.9757	0.9991	0.9991	0.9999	0.9866
22	0.9992	0.9999	0.9933	0.9998	0.9998	0.9999	0.9774
23	0.9960	0.9996	0.9283	0.9930	0.9966	0.9961	0.9937
24	0.9910	0.9980	0.9574	0.9952	0.9893	0.9964	0.9613
25	0.9985	0.9985	0.9351	0.9960	0.9891	0.9988	0.9608
26	0.9561	0.9894	0.8026	0.9966	0.9799	0.9964	0.9916
27	0.9948	0.9995	0.9721	0.9979	0.9967	0.9979	0.9923
28	0.9787	0.9886	0.9483	0.9988	0.9752	0.9961	0.9905
29	0.9104	0.9987	0.8382	0.9707	0.9985	0.9948	0.9986
30	0.9738	0.9992	0.9429	0.9831	0.9992	0.9961	0.9925

Note: data for projectiles 1-20 are taken from [12] (average values for minimum 5 tests for each projectile); results for projectiles 21-23 are from [7], projectiles 24-26 are from [9], 27-28 from [3] and 29-30 from [5] (test cylinders)

CONCLUSION

The analysis of the most relevant theoretical fragment mass distribution models has been undertaken. Using the statistical approach based on comparison with 30 projectiles, it has been concluded that the generalized Mott, the generalized Grady and the Weibull distribution give a very good description of experimental data. The slight advantage of the generalized Grady distribution can be physically justified. The suggested distribution models can be applied in HE projectile efficiency modeling.

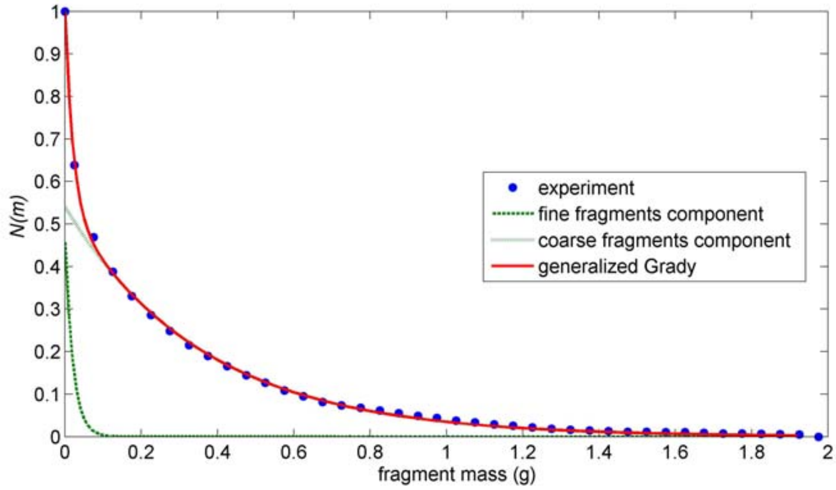

Figure 3. Fine and coarse fragments components of the generalized Grady distribution fit of the experimental data (projectile 1[12])

REFERENCES

[1] N.F. Mott, A theory of fragmentation, *British Ministry of Supply Repts*, 3348, 3642, 4035, (1943)

[2] D.E. Grady, et al, Comparing Alternate Approaches in the Scaling of Naturally Fragmenting Munitions, *19th International Symposium of Ballistics*, Interlaken, pp. 591-597, (2001)

[3] J. Dehn, Probability formulas for describing fragment size distribution, *Ballistic research laboratory, Abredeen Proving Ground*, ARBRL-TR-02332, (1981)

[4] D.E. Grady, M.E. Kipp, Geometric statistics and dynamic fragmentation, *J. App. Phys.*, **58** (3), pp. 1210-1222, (1985)

[5] E.A. Cohen, New Formulas for Predicting the Size Distribution of Warhead Fragments, *Mathematical Modelling*, **2**, pp. 19-32, (1981)

[6] L. Baker, et al, Fracture and Spall Ejecta Mass Distribution: Lognormal and Multifractal Distributions, *J. Appl. Phys.*, **72** (7), pp. 2724-2731, (1992)

[7] M. Held, Fragment Mass Distribution of HE Projectile, *Prop., Explos., Pyrotech.*, **15**, pp. 254-260, (1990)

[8] P. Elek, S. Jaramaz, Fragment Mass Distribution Laws for HE Projectiles (in Serbian), *XXII Symposium for explosive materials JKEM*, Bar, pp. 241-251, (2004)

[9] Strømsøe, E., Ingebrigtsen, K.O., A Modification of the Mott Formula for Prediction of the Fragment Size Distribution, *Prop., Explos., Pyrotech.*, **12**, pp. 175-178, (1987)

[10] H. Inaoka, M. Ohno, New Universality Class of Impact Fragmentation, *Fractals*, **11** (4), pp. 369-376, (2003)

[11] J.J. Gilvarry, Fracture of Brittle Solids I. Distribution Function for Fragment Size for Single Fracture (Theoretical), *J. Appl. Phys.*, **32**, pp. 391-399, (1961)

[12] R. Jovanović, Evaluation of regularity of HE projectiles fragmentation using statistical tests, MSc Thesis, *Faculty of Mechanical Engineering*, Belgrade, (2002)